

Intermediate Coherent-entangled State Representation: Generation and its applications

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By combining the beam splitter and the Fresnel transform, a protocol is proposed to generate a new entangled state representation, called the intermediate coherent-entangled state (ICES) representation. The properties, such as eigenvalue equation, completeness relation and orthogonal relation, are investigated. The conjugate state representation of the ICES and the Schmidt decomposing of the ICES are also discussed. As applications, a new squeezing operator and some operator identities by using the ICES are obtained.

I. INTRODUCTION

The quantum mechanical representation theory proposed by Dirac [1] has made a profound effect on the development of modern quantum physics. The symbolic method combined with the technique of integration within an ordered product (IWOP) of operators has been found to be a useful method to find the new quantum mechanical representation [2–4]. In recent years, entangled state representations [5] have played a more important role in quantum optics [6, 7], quantum information and quantum computation [8–11].

Actually, quantum mechanical representation can construct a bridge between classical linear integral transform and quantum unitary operator [12]. That is to say, different linear integral transforms can be described by different quantum operators, such as the generalized Fresnel transform [13] in 1-dimensional (1D) can be described by the 1-mode Fresnel operator $F_1(r, s)$ [14] and the 2-mode Fresnel transform can be described by 2-mode Fresnel operator $F_2(r, s)$ [15] in quantum optics. These quantum operators acting on quantum states can generate new quantum mechanical representations. For instance, Fresnel operator $F_1(r, s)$ acting on the coordinate eigenstate $|q\rangle$ can generate the intermediate coordinate-momentum representation $|q\rangle_{s,r}$ [16], and Fresnel operator $F_2(r, s)$ applying to the entangled state $|\eta\rangle$ [17] can generate the intermediate entangled state representation $|\eta\rangle_{s,r}$ [2, 16].

On the other hand, beam splitter (BS) has played an important role in generating entangled states, such as a three-mode continuous-variables (CV) entangled state was proposed by using an asymmetric BS [18–21] and a parametric down-conversion (PDC) instrument, and then a four-mode CV entangled state was generated by BS and PDCs [22]. In addition, multi-mode CV entangled state was generated by a BS and a polarizer or by BS, PDC and a polarizer [19, 23, 24]. It is no doubt that all these studies above exhibit the role of BS in generating entangled states and indicate that these physical instruments, such as PDC, BS and polarizer, can be used to implement entangled states.

Then an interesting question thus naturally arises: is there any kind of entangled state which can be generated by combining BS and Fresnel transform optical process? As far as our knowledge goes, there has no report in the literature up to now. In this paper, we will introduce a new entangled state representation which is called the intermediate coherent-entangled state (ICES) representation. It is shown that it can be generated by the BS and the Fresnel transform.

The paper is arranged as follows. In Section 2, we introduce the Fresnel operator and the intermediate coordinate-momentum state (ICMS) representations which can be obtained by the Fresnel operator [12]. In section 3, we present two methods to produce ICES, which is a new entangled state representation and is not been presented before. Deriving the ICES representation naturally from the completeness relations of the coherent state and the generalized coordinate state by IWOP technique is one method. Producing the ICES by combining a symmetric BS and the Fresnel transformation is another method. In section 4, we discuss the characters of the ICES, including the eigenvalue equation, orthogonal relation, Schmidt decomposition and conjugate state. As the applications of the new representation, in section 5, we derive a new squeezing operator and derive some operator identities. We end our work in section 6. These discussions exhibit not only the usefulness of the ICES representation, but also provide a considerable insight into the character of them.

II. FRESNEL OPERATOR AND THE ICMS

In Ref.[16, 25–28], the Fresnel operator $F_1(r, s)$ is proposed by using the coherent state representation $|z\rangle = \exp\left[-\frac{1}{2}|z|^2 + za^\dagger\right]|0\rangle$ and the IWOP technique [2], which is given by

$$F_1(r, s) = \frac{1}{\sqrt{s^*}} \exp\left(\frac{r}{-2s^*} a^{\dagger 2}\right) : \exp\left[\left(\frac{1}{s^*} - 1\right) a^\dagger a\right] : \exp\left(\frac{r^*}{2s^*} a^2\right), \quad (1)$$

where the symbol $:$ denotes the normally ordering, and s and r are complex and satisfy the unimodularity condition $ss^* - rr^* = 1$. The Fresnel operator $F_1(r, s)$ corresponds to optical Fresnel transformation characteristic of ray transfer matrix elements (A, B, C, D) , $AD - BC = 1$, connecting the input light field $f(x)$ and output light field $g(x)$ by the Fresnel integration [15],

$$g(x') = \frac{1}{\sqrt{2\pi i B}} \int_{-\infty}^{\infty} \exp\left[\frac{i}{2B}(Ax^2 - 2x'x + Dx'^2)\right] f(x) dx. \quad (2)$$

Parameters r and s in Eq.(1) are related to $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ transfer matrix by

$$\begin{aligned} s &= \frac{1}{2}[(A + D) - i(B - C)], \\ r &= -\frac{1}{2}[(A - D) + i(B + C)]. \end{aligned} \quad (3)$$

Let the Fresnel operator $F_1(r, s)$ act on the coordinate eigenstate $|q\rangle$, and use the completeness relation of the coherent state $\int \frac{d^2 z}{\pi} |z\rangle \langle z| = 1$, we obtain the ICMS $|q\rangle_{s,r}$ [16],

$$\begin{aligned} F_1(r, s) |q\rangle &= \frac{\pi^{-1/4}}{\sqrt{s^* + r^*}} \exp\left[\frac{r^* - s^*}{2(s^* + r^*)} q^2 + \frac{\sqrt{2}q}{s^* + r^*} a^\dagger - \frac{(s + r)}{2(s^* + r^*)} a^{\dagger 2}\right] |0\rangle \\ &\equiv |q\rangle_{s,r}, \end{aligned} \quad (4)$$

where we used the following relations,

$$\langle z | q \rangle = \pi^{-1/4} \exp\left\{-\frac{q^2}{2} - \frac{|z|^2}{2} + \sqrt{2}qz^* - \frac{z^{*2}}{2}\right\}, \quad (5)$$

and the integration formula

$$\begin{aligned} &\int \frac{d^2 z}{\pi} \exp\left(\zeta |z|^2 + \xi z + \eta z^* + fz^2 + gz^{*2}\right) \\ &= \frac{1}{\sqrt{\zeta^2 - 4fg}} \exp\left[\frac{-\zeta\xi\eta + \xi^2 g + \eta^2 f}{\zeta^2 - 4fg}\right], \end{aligned} \quad (6)$$

whose convergent condition is $\text{Re}(\zeta \pm f \pm g) < 0$, $\text{Re}[(\zeta^2 - 4fg) / (\zeta \pm f \pm g)] < 0$. According to Eq.(3), we also have

$$|q\rangle_{s,r} = \frac{\pi^{-1/4}}{\sqrt{D + iB}} \exp\left[-\frac{A - iC}{D + iB} \frac{q^2}{2} + \frac{\sqrt{2}q}{D + iB} a^\dagger - \frac{D - iB}{D + iB} \frac{a^{\dagger 2}}{2}\right] |0\rangle. \quad (7)$$

Correspondingly, the eigenvalue equation of $|q\rangle_{s,r}$ is

$$(DQ - BP) |q\rangle_{s,r} = q |q\rangle_{s,r}, \quad (8)$$

where $Q = (a + a^\dagger)/\sqrt{2}$, $P = (a - a^\dagger)/(i\sqrt{2})$. When $A = D = 1, B = C = 0$, the above equation evolves into $Q |q\rangle_{s,r} = q |q\rangle_{s,r}$, and $|q\rangle_{s,r}$ is just the coordinate operator eigenstate $|q\rangle$. When $A = D = 0, -B = C = 1$, the

above equation evolves into $P|q\rangle_{s,r} = q|q\rangle_{s,r}$, and $|q\rangle_{s,r}$ is just the momentum operator eigenstate $|p = q\rangle$. So $|q\rangle_{s,r}$ is named intermediate coordinate-momentum state.

In a similar way, operating the Fresnel operator on the momentum eigenstate $|p\rangle$, together with the completeness relation of the coherent state, we obtain the intermediate momentum-coordinate state (IMCS) $|p\rangle_{s,r}$,

$$\begin{aligned} F_1(r, s) |p\rangle &= \frac{\pi^{-1/4}}{\sqrt{s^* - r^*}} \exp \left[\frac{\sqrt{2}ip}{s^* - r^*} a^\dagger + \frac{s - r}{2(s^* - r^*)} a^{\dagger 2} - \frac{s^* + r^*}{2(s^* - r^*)} p^2 \right] |0\rangle \\ &\equiv |p\rangle_{s,r}, \end{aligned} \quad (9)$$

or

$$|p\rangle_{s,r} = \frac{\pi^{-1/4}}{\sqrt{A - iC}} \exp \left[-\frac{D + iB}{A - iC} \frac{p^2}{2} + \frac{\sqrt{2}ip}{A - iC} a^\dagger + \frac{A + iC}{A - iC} \frac{a^{\dagger 2}}{2} \right] |0\rangle. \quad (10)$$

Correspondingly, the eigenvalue equation of $|p\rangle_{s,r}$ is

$$(AP - CQ) |p\rangle_{s,r} = p |p\rangle_{s,r}, \quad (11)$$

so $|p\rangle_{s,r}$ is the eigenvector of operator $AP - CQ$. When $A = D = 1, B = C = 0$, the above equation evolves into $P|p\rangle_{s,r} = p|p\rangle_{s,r}$, and $|p\rangle_{s,r}$ is just the momentum operator eigenstate $|p\rangle$. When $A = D = 0, -B = C = 1$, the above equation evolves into $Q|p\rangle_{s,r} = -p|p\rangle_{s,r}$, and $|p\rangle_{s,r}$ is just the coordinate operator eigenstate $|x = -p\rangle$. So it is named the IMCS representation. $|q\rangle_{s,r}$ and $|p\rangle_{s,r}$ will be used to construct a new entangled state representation in the following discussion.

III. THE INTERMEDIATE COHERENT-ENTANGLED STATE (ICES)

In this section, we propose the ICES representation and its generating protocol.

A. The ICES obtained by IWOP technique

First, let us construct the ICES by using the IWOP technique. It is well known that the completeness relation of the coherent state $|z\rangle$ is $\int \frac{d^2z}{\pi} |z\rangle \langle z| = \int \frac{d^2z}{\pi} : \exp\{-(z - a)(z^* - a^\dagger)\} : = 1$, which can be rewritten as

$$1 = \int \frac{d^2z}{\pi} : \exp \left\{ - \left(z - \frac{a + b}{\sqrt{2}} \right) \left(z^* - \frac{a^\dagger + b^\dagger}{\sqrt{2}} \right) \right\} :, \quad (12)$$

where a and b are two bosonic subtraction operators. The completeness relation of the coordination state $|q\rangle$ is

$$1 = \int_{-\infty}^{\infty} dq |q\rangle \langle q| = \int_{-\infty}^{\infty} \frac{dq}{\sqrt{\pi}} : e^{-(q - Q)^2} :, \quad (13)$$

where $Q = \frac{a + a^\dagger}{\sqrt{2}}$ is the coordinate operator. If we make the following transform about a^\dagger and a in (13),

$$a^\dagger \rightarrow \frac{s + r}{\sqrt{2}} (b^\dagger - a^\dagger), a \rightarrow \frac{s^* + r^*}{\sqrt{2}} (b - a), \quad (14)$$

i.e.

$$a^\dagger \rightarrow \frac{(D - iB)}{\sqrt{2}} (b^\dagger - a^\dagger), a \rightarrow \frac{(D + iB)}{\sqrt{2}} (b - a), \quad (15)$$

then we have the completeness relation of the generalized coordinate state,

$$\frac{\pi^{-1/2}}{\sqrt{D^2 + B^2}} \int_{-\infty}^{\infty} dq : \exp \left\{ - \frac{\left[q - \frac{(D-iB)(b^\dagger - a^\dagger) + (D+iB)(b-a)}{2} \right]^2}{D^2 + B^2} \right\} : = 1. \quad (16)$$

Then combining (12) and (16), we obtain the following completeness relation

$$\begin{aligned} 1 &= \frac{\pi^{-1/2}}{\sqrt{D^2 + B^2}} \int_{-\infty}^{\infty} dq \int \frac{d^2 z}{\pi} \\ &\times : \exp \left\{ - \left(z - \frac{a+b}{\sqrt{2}} \right) \left(z^* - \frac{a^\dagger + b^\dagger}{\sqrt{2}} \right) \right\} \\ &\times \exp \left\{ - \frac{\left[q - \frac{(D-iB)(b^\dagger - a^\dagger) + (D+iB)(b-a)}{2} \right]^2}{D^2 + B^2} \right\} : . \end{aligned} \quad (17)$$

Moreover, by using IWOP technique and the normal product form of the two-mode vacuum projector $|00\rangle\langle 00| = : \exp[-a^\dagger a - b^\dagger b] : ,$ the above equation (17) can be decomposed as

$$1 = \int_{-\infty}^{\infty} dq \int \frac{d^2 z}{\pi} |z, q\rangle_{s,r} \langle z, q|, \quad (18)$$

where $|z, q\rangle_{s,r}$ is called as intermediate coherent-entangled state (ICES) and it reads as

$$\begin{aligned} |z, q\rangle_{s,r} &= \frac{\pi^{-1/4}}{\sqrt{D+iB}} \exp \left\{ - \frac{|z|^2}{2} - \frac{A-iC}{D+iB} \frac{q^2}{2} + \left(\frac{z}{\sqrt{2}} + \frac{q}{D+iB} \right) b^\dagger \right\} \\ &\exp \left\{ \left(\frac{z}{\sqrt{2}} - \frac{q}{D+iB} \right) a^\dagger - \frac{D-iB}{D+iB} \frac{(b^\dagger - a^\dagger)^2}{4} \right\} |00\rangle \equiv |\zeta\rangle. \end{aligned} \quad (19)$$

which is the new entangled state representation. The completeness relation of $|\zeta\rangle$ is just Eq.(18).

B. Generation of the ICES by using a symmetric BS and Fresnel transform (FT)

Next, we introduce the scheme of producing ICES through the BS and FT as showed in Fig.1. The BS plays important role in generating some new quantum states. For example, a variable arcsine state can be generated through a variable BS [29], a coherent-entangled state can be generated by asymmetric BS [20]. In Fig.1, the input a -mode is coherent state $|z\rangle_a$, the input b -mode is ICMS $|q\rangle_{b,s,r}$, the two input modes are two input ports of the 50:50 BS. The initial state of the system is

$$|\Phi\rangle_{in} = |z\rangle_a \otimes |q\rangle_{b,s,r}, \quad (20)$$

where

$$|q\rangle_{b,s,r} = \frac{\pi^{-1/4}}{\sqrt{D+iB}} \exp \left\{ - \frac{A-iC}{D+iB} \frac{q^2}{2} + \frac{\sqrt{2}q}{D+iB} b^\dagger - \frac{D-iB}{D+iB} \frac{b^{\dagger 2}}{2} \right\} |0\rangle_b. \quad (21)$$

According to Ref.[30], we know the beam splitter matrix transformation between the input operators (a^\dagger, b^\dagger) and the output operators (c^\dagger, d^\dagger),

$$\begin{pmatrix} c^\dagger \\ d^\dagger \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} a^\dagger \\ b^\dagger \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a^\dagger \\ b^\dagger \end{pmatrix}, \quad (22)$$

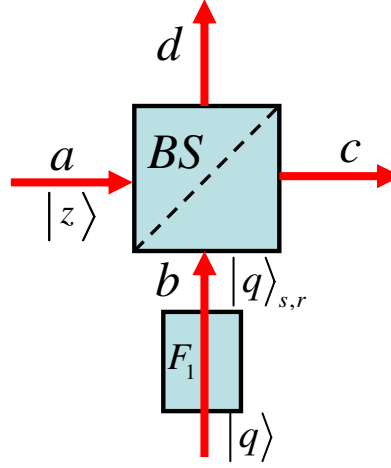


FIG. 1: (Color online) The project to produce the intermediate coherent-entangled state $|\zeta\rangle$.

i.e.

$$a^\dagger \rightarrow c^\dagger = \frac{1}{\sqrt{2}} (a^\dagger + b^\dagger), b^\dagger \rightarrow d^\dagger = \frac{1}{\sqrt{2}} (b^\dagger - a^\dagger). \quad (23)$$

In fact, corresponding to the input and output process of the beam, the transform process of the quantum states can be regarded as the process of an unitary operator ($B(\theta)$) acting on a quantum state, $B(\frac{\pi}{4}) |z\rangle_a |q\rangle_{b,s,r}$. According to Ref.[12, 16, 31], the BS operator $B(\theta) = e^{\theta(ab^\dagger - a^\dagger b)}$ satisfy the following relations

$$\begin{aligned} B(\theta) a^\dagger B^\dagger &= a^\dagger \cos \theta + b^\dagger \sin \theta, \\ B(\theta) b^\dagger B^\dagger &= b^\dagger \cos \theta - a^\dagger \sin \theta, \\ B^\dagger &= B(-\theta) = B^{-1}. \end{aligned} \quad (24)$$

After the beam passing through the BS, the output state reads $|\Phi\rangle_{out} = B(\theta) |\Phi\rangle_{in}$, i.e.

$$\begin{aligned} |\Phi\rangle_{out} &= B\left(\frac{\pi}{4}\right) |z\rangle_a \otimes |q\rangle_{b,s,r} \\ &= \frac{\pi^{-1/4}}{\sqrt{D+iB}} \exp \left\{ -\frac{A-iC}{D+iB} \frac{q^2}{2} + \frac{q}{D+iB} (b^\dagger - a^\dagger) \right. \\ &\quad \left. - \frac{D-iB}{D+iB} \frac{(b^\dagger - a^\dagger)^2}{4} - \frac{1}{2} |z|^2 + z \frac{(a^\dagger + b^\dagger)}{\sqrt{2}} \right\} |00\rangle \\ &\equiv |z, q\rangle_{s,r} = |\zeta\rangle, \end{aligned} \quad (25)$$

where we have used relations (24). This entangled state is just the ICES $|\zeta\rangle$. The whole project to produce the ICES is showed as follow, the first light route (route a) is coherent state $|z\rangle_a$, the second light route (route b) is the ICMS which can be obtained by the Fresnel transformation (F_1) from a light in coordinate state passing through the nonlinear optical process. Then after the two routes of light passing through the BS (linear optical apparatus), we obtain the new entangle state ICES. So we present a new project to produce the entangled state by mixing the BS optical apparatus and the Fresnel transformation optical process. Thus the scheme of producing the ICES is brought forward theoretically.

IV. THE CHARACTERS OF THE ICES

In this section, we shall examine the characters of the ICES, including the eigenvalue equation, orthogonal relation and Schmidt decomposition.

A. The eigenvalue equation of $|\zeta\rangle$

Let annihilate operator a act on the state $|\zeta\rangle$, we have

$$\left[a - \frac{D - iB}{D + iB} \frac{b^\dagger}{2} + \frac{D - iB}{D + iB} \frac{a^\dagger}{2} \right] |\zeta\rangle = \left[\frac{z}{\sqrt{2}} - \frac{q}{D + iB} \right] |\zeta\rangle, \quad (26)$$

$$\left[b + \frac{D - iB}{D + iB} \frac{b^\dagger}{2} - \frac{D - iB}{D + iB} \frac{a^\dagger}{2} \right] |\zeta\rangle = \left[\frac{z}{\sqrt{2}} + \frac{q}{D + iB} \right] |\zeta\rangle, \quad (27)$$

Combining Eq.(26) and (27), we obtain the eigenvalue equations of $|\zeta\rangle$, i.e.,

$$\begin{aligned} (a + b) |\zeta\rangle &= \sqrt{2}z |\zeta\rangle, \\ [D(Q_b - Q_a) - B(P_b - P_a)] |\zeta\rangle &= \sqrt{2}q |\zeta\rangle, \end{aligned} \quad (28)$$

where $Q_a = (a + a^\dagger)/\sqrt{2}$, $Q_b = (b + b^\dagger)/\sqrt{2}$, $P_a = (a - a^\dagger)/(i\sqrt{2})$, $P_b = (b - b^\dagger)/(i\sqrt{2})$. It is obvious that operators $D(Q_b - Q_a) - B(P_b - P_a)$ and $a + b$ are commutative, i.e., $[D(Q_b - Q_a) - B(P_b - P_a), a + b] = 0$, thus the common eigenvector of the two operators is $|\zeta\rangle$, which is actually an entangled state. Especially, when $A = D = 1$, $B = C = 0$, the eigenvalue equation become $(Q_b - Q_a) |\zeta\rangle = \sqrt{2}q |\zeta\rangle$, $(a + b) |\zeta\rangle = \sqrt{2}z |\zeta\rangle$. $|\zeta\rangle$ just reduces to the coherent entangled state $|z, q\rangle$. When $A = D = 0$, $-B = C = 1$, the above state $|\zeta\rangle$ turns to be another coherent entangled state $|z, p\rangle$, whose eigenvalue equation is $(P_b - P_a) |\zeta\rangle = \sqrt{2}q |\zeta\rangle$, $(a + b) |\zeta\rangle = \sqrt{2}z |\zeta\rangle$.

B. Orthogonality of $|\zeta\rangle$

Next, we shall use Eq.(25) to obtain the orthogonality of $|\zeta\rangle$. Notice that $|\zeta\rangle = B|z\rangle_a \otimes |q\rangle_{b,s,r}$ and $|q\rangle_{b,s,r} = F_1|q\rangle_b$, we can express $\langle\zeta'|\zeta\rangle$ as

$$\begin{aligned} \langle\zeta'|\zeta\rangle &= {}_{s,r}\langle z', q' | z, q \rangle_{s,r} \\ &= {}_b\langle q' | \otimes {}_a\langle z' | F_1^\dagger B^\dagger B F_1 | z \rangle_a \otimes | q \rangle_b \\ &= \delta(q - q') \langle z' | z \rangle \\ &= \delta(q - q') \exp \left[z'^* z - \frac{1}{2} (|z'|^2 + |z|^2) \right], \end{aligned} \quad (29)$$

where we use the relations $B^\dagger B = 1$ and $F_1^\dagger F_1 = 1$, as well as the orthogonal relation of coordinate state $\langle q' | q \rangle = \delta(q - q')$. Especially, when $z = z'$, we obtain ${}_{s,r}\langle z', q' | z, q \rangle_{s,r} = \delta(q - q')$, so $|z, q\rangle_{s,r}$ is a partially orthogonal state, and this character is similar to the coherent state.

C. The Schmidt decomposition of $|\zeta\rangle$

In order to see the entanglement property of the ICES, here we consider the Schmidt decomposing of $|\zeta\rangle$. Using the coordinate representation of BS operator [32],

$$B\left(\frac{\pi}{4}\right) = \int_{-\infty}^{\infty} dq_1 dq_2 \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right|, \quad (30)$$

where $\left| \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right\rangle \equiv |q_1\rangle \otimes |q_2\rangle$, we can obtain the Schmidt decomposing of $|\zeta\rangle$,

$$\begin{aligned} |\zeta\rangle &= B\left(\frac{\pi}{4}\right) |z\rangle_a |q\rangle_{b,s,r} \\ &= \frac{1}{\pi^{3/4} \sqrt{2iB}} e^{-\frac{|z|^2}{2} - \frac{z^2}{2}} \int_{-\infty}^{\infty} dq_1 dq_2 \exp \left(\sqrt{2}q_1 z - \frac{1}{2}q_1^2 + \frac{2q_2 q - Dq_2^2 - Aq^2}{2iB} \right) \\ &\quad \times \left| \frac{1}{\sqrt{2}} (q_1 - q_2) \right\rangle \otimes \left| \frac{1}{\sqrt{2}} (q_1 + q_2) \right\rangle, \end{aligned} \quad (31)$$

where we used the following relations

$$\begin{aligned}\langle q_1 | z \rangle_a &= \pi^{-1/4} \exp \left\{ -\frac{q_1^2}{2} - \frac{|z|^2}{2} + \sqrt{2} q_1 z - \frac{z^2}{2} \right\}, \\ \langle q_2 | q \rangle_{b,s,r} &= \frac{\pi^{-1/2}}{\sqrt{2iB}} \exp \left[\frac{2q_2 q - Dq_2^2 - Aq^2}{2iB} \right].\end{aligned}\quad (32)$$

Let $\frac{q_1 - q_2}{\sqrt{2}} = q'$, then $q_1 = \sqrt{2}q' + q_2$. Substituting the expression of q_1 into (31), we obtain

$$\begin{aligned}|\zeta\rangle &= \frac{1}{\pi^{3/4} \sqrt{iB}} \int_{-\infty}^{\infty} dq' dq_2 |q'\rangle_a \otimes \left| \left(q' + \sqrt{2}q_2 \right) \right\rangle_b \exp \left(\frac{2q_2 q - Dq_2^2 - Aq^2}{2iB} \right) \\ &\times \exp \left[-\frac{|z|^2}{2} + \left(2q' + \sqrt{2}q_2 \right) z - \frac{z^2}{2} - \frac{1}{2} \left(\sqrt{2}q' + q_2 \right)^2 \right],\end{aligned}\quad (33)$$

which is the Schmidt decomposing of $|\zeta\rangle$ in coordinate representation. Thus $|\zeta\rangle$ is just an entangled state.

D. Conjugate state of $|\zeta\rangle$

We can also obtain the conjugate state $|\kappa\rangle$ of $|\zeta\rangle$ by considering the input mode a as coherent state $|z\rangle$, the input mode b as the IMCS $|p\rangle_{s,r}$. Considering that the input a -mode is coherent state $|z\rangle$, the input b -mode is the IMCS $|p\rangle_{s,r}$, the two input modes are two input ports of the Beam splitter (BS), the ratio between the reflection coefficient and the transmission coefficient is 50: 50. The initial state of the system is $|\Phi\rangle_{in} = |z\rangle_a \otimes |p\rangle_{b,s,r}$, where

$$|p\rangle_{b,s,r} = \frac{\pi^{-1/4}}{\sqrt{A-iC}} \exp \left[-\frac{D+iB}{2(A-iC)} p^2 + \frac{\sqrt{2}ip}{A-iC} b^\dagger + \frac{A+iC}{2(A-iC)} b^{\dagger 2} \right] |0\rangle. \quad (34)$$

After the beam passing through the BS, in a similar way we have

$$\begin{aligned}|\kappa\rangle &= B\left(\frac{\pi}{4}\right) |z\rangle_a \otimes |p\rangle_{b,s,r} \\ &= \frac{\pi^{-1/4}}{\sqrt{A-iC}} \exp \left\{ -\frac{|z|^2}{2} + z \frac{(a^\dagger + b^\dagger)}{\sqrt{2}} \right\} \\ &\times \exp \left\{ -\frac{D+iB}{A-iC} \frac{p^2}{2} + \frac{ip}{A-iC} (b^\dagger - a^\dagger) + \frac{A+iC}{(A-iC)} \frac{(b^\dagger - a^\dagger)^2}{4} \right\} |0\rangle_a |0\rangle_b.\end{aligned}\quad (35)$$

This new representation is another kind of the intermediate coherent-entangled state $|\kappa\rangle$. The eigenvalue equation of the state $|\kappa\rangle$ are

$$\begin{aligned}(b+a)|\kappa\rangle &= \sqrt{2}z|\kappa\rangle, \\ [A(P_b - P_a) - C(Q_b - Q_a)]|\kappa\rangle &= \sqrt{2}p|\kappa\rangle.\end{aligned}\quad (36)$$

Thus $|\kappa\rangle$ is the common eigenvector of $a+b$ and $A(P_b - P_a) - C(Q_b - Q_a)$. We can easily obtain $[A(P_b - P_a) - C(Q_b - Q_a), D(Q_b - Q_a) - B(P_b - P_a)] = -2i\hbar(AD - BC) = -2i\hbar$, so $|\kappa\rangle$ is the conjugate state of $|\zeta\rangle$.

V. APPLICATIONS OF THE ICES

A. To obtain the squeezing operator and its exponential form

As an important application of $|z, q\rangle_{s,r}$, we derive a new squeezing operator. By constructing the following ket-bra integration,

$$U = \frac{1}{\sqrt{\mu}} \int_{-\infty}^{\infty} dq \int \frac{d^2 z}{\pi} \left| z, \frac{q}{\mu} \right\rangle_{r,sr,s} \langle z, q|, \quad (37)$$

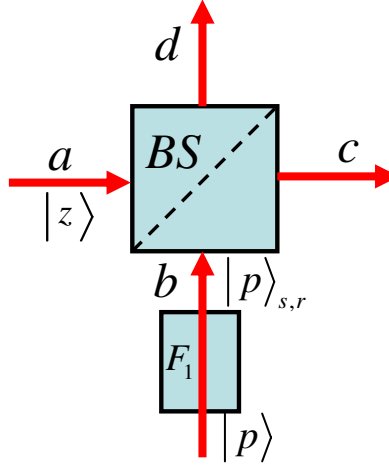


FIG. 2: (Color online) The project to produce the intermediate coherent-entangled state $|\kappa\rangle$.

where $\mu = e^\lambda$, $\left|z, \frac{q}{\mu}\right\rangle_{s,r} = B|z\rangle_a \otimes \left|\frac{q}{\mu}\right\rangle_{b,s,r}$, $\left|\frac{q}{\mu}\right\rangle_{b,s,r} = F_1 \left|\frac{q}{\mu}\right\rangle_b$, and $\left|\frac{q}{\mu}\right\rangle_b = \sqrt{\mu} S_1 |q\rangle_b$, in which $S_1(\lambda) = e^{\frac{\lambda}{2}(b^2 - b^{\dagger 2})}$ is the single mode squeezing operator, F_1 is the Fresnel operator which acts on mode b only, $B(\frac{\pi}{4})$ is the BS operator which acts on mode a and mode b simultaneously. Thus we have

$$\begin{aligned} \left|z, \frac{q}{\mu}\right\rangle_{s,r} &= B F_1 |z\rangle_a \otimes \left|\frac{q}{\mu}\right\rangle_b = \sqrt{\mu} B F_1 S_1 |z\rangle_a |q\rangle_b, \\ \langle z, q|_{r,s} &= {}_a \langle z|_b \langle q| F_1^\dagger B^\dagger, \end{aligned} \quad (38)$$

Using the following relations

$$\begin{aligned} F_1(s, r) b F_1^\dagger(s, r) &= s^* b + r b^\dagger, F_1^\dagger(s, r) b F_1(s, r) = s b - r b^\dagger, \\ F_1(s, r) b^\dagger F_1^\dagger(s, r) &= s b^\dagger + r^* b, F_1^\dagger(s, r) b^\dagger F_1(s, r) = s^* b^\dagger - r^* b, \\ F_1^\dagger &= F_1(r \rightarrow -r, s \rightarrow s^*), \\ S_1 b S_1^\dagger &= b \cosh \lambda + b^\dagger \sinh \lambda, \end{aligned} \quad (39)$$

and the completeness relations ($\int_{-\infty}^{\infty} dq \int \frac{d^2 z}{\pi} |z, q\rangle \langle z, q| = 1$), we obtain

$$\begin{aligned} U &= \frac{1}{\sqrt{\mu}} \int_{-\infty}^{\infty} dq \int \frac{d^2 z}{\pi} \left|z, \frac{q}{\mu}\right\rangle_{s,r,s,r} \langle z, q| \\ &= B F_1 S_1 \int_{-\infty}^{\infty} dq \int \frac{d^2 z}{\pi} |z, q\rangle \langle z, q| F_1^\dagger B^\dagger \\ &= B F_1 S_1 F_1^\dagger B^\dagger \\ &= \exp \left\{ \frac{\lambda}{2} \left[\frac{1}{2} (s^{*2} - r^{*2}) (a - b)^2 \right] \right. \\ &\quad \left. + \frac{\lambda}{2} [s^* r (1 + b^\dagger b + a^\dagger a - a^\dagger b - a b^\dagger)] - c.c \right\}, \end{aligned} \quad (40)$$

where $c.c$ represents Hermitian conjugate. It is obvious that $U^\dagger = U(-\lambda) = U^{-1}$ is an unitary operator. The last equation of Eq.(40) is the exponential form of unitary U . Using the Baker-Hausdroff formula $e^{\xi A} B e^{-\xi A} = B + \xi [A, B] + \frac{\xi^2}{2!} [A, [A, B]] + \dots$, we can get

$$\begin{aligned} U a U^\dagger &= \frac{1}{2} [(b - a) [(r s^* - s r^*) \sinh \lambda - \cosh \lambda] \\ &\quad + (b^\dagger - a^\dagger) (r^2 - s^2) \sinh \lambda + (a + b)], \\ U b U^\dagger &= \frac{1}{2} [(b - a) [\cosh \lambda - (r s^* - s r^*) \sinh \lambda] \\ &\quad + (b^\dagger - a^\dagger) (s^2 - r^2) \sinh \lambda + (a + b)], \end{aligned} \quad (41)$$

where we have used Eq. (24) and the relation $ss^* - rr^* = 1$. It then follows that

$$\begin{aligned} UQ_aU^\dagger &= \frac{1}{2}\{(Q_a + Q_b) + i(P_a - P_b) \frac{(r-s^*)^2 - (r^*-s)^2}{2} \sinh \lambda \\ &\quad + (Q_a - Q_b) [\cosh \lambda - \frac{r^2 - s^2 + r^{*2} - s^{*2}}{2} \sinh \lambda]\}, \\ UQ_bU^\dagger &= \frac{1}{2}\{(Q_a + Q_b) - i(P_a - P_b) \frac{(r-s^*)^2 - (r^*-s)^2}{2} \sinh \lambda \\ &\quad + (Q_b - Q_a) [\cosh \lambda - \frac{r^2 - s^2 + r^{*2} - s^{*2}}{2} \sinh \lambda]\}, \end{aligned} \quad (42)$$

and

$$\begin{aligned} UP_aU^\dagger &= \frac{1}{2}\{(P_a + P_b) + (Q_b - Q_a) \frac{(r+s^*)^2 - (r^*+s)^2}{2i} \sinh \lambda \\ &\quad + (P_b - P_a) [\frac{s^{*2} - r^{*2} - r^2 + s^2}{2} \sinh \lambda - \cosh \lambda]\}, \\ UP_bU^\dagger &= \frac{1}{2}\{(P_a + P_b) + (Q_b - Q_a) \frac{(r^*+s)^2 - (r+s^*)^2}{2i} \sinh \lambda \\ &\quad + (P_b - P_a) [\frac{-s^{*2} + r^{*2} - s^2 + r^2}{2} \sinh \lambda + \cosh \lambda]\}. \end{aligned} \quad (43)$$

Furthermore, combining Eq. (42) and (43), we obtain

$$\begin{aligned} U(Q_a + Q_b)U^\dagger &= Q_a + Q_b, \\ U(Q_a - Q_b)U^\dagger &= \{i(P_a - P_b) \frac{(r-s^*)^2 - (r^*-s)^2}{2} \sinh \lambda \\ &\quad + (Q_a - Q_b) [\cosh \lambda - \frac{(r^2 - s^2 + r^{*2} - s^{*2})}{2} \sinh \lambda]\}, \\ U(P_a + P_b)U^\dagger &= P_a + P_b, \\ U(P_a - P_b)U^\dagger &= \{-i(Q_b - Q_a) \frac{(r+s^*)^2 - (r^*+s)^2}{2} \sinh \lambda \\ &\quad + (P_b - P_a) [\frac{s^{*2} - r^{*2} - r^2 + s^2}{2} \sinh \lambda - \cosh \lambda]\}. \end{aligned} \quad (44)$$

When $s = 1, r = 0$, the above results just degenerate to the case of two-mode squeezing operator $U = \frac{1}{\sqrt{\mu}} \int_{-\infty}^{\infty} dq \int \frac{d^2 z}{\pi} \left| z, \frac{q}{\mu} \right\rangle \langle z, q|$ (see reference [19]). If we define $a'^\dagger = Ua^\dagger U^\dagger$, we can verify $[a', a'^\dagger] = 1$.

B. To derive some operator identities

In quantum mechanics, quantum operators are usually non-commuting, as many as possible operator identities are searched to apply in all kinds of physical problems. By using the intermediate coherent-entangled state $|\zeta\rangle$ and the IWOP technique, we can construct some operator identities, such as $e^{D(Q_b - Q_a) - B(P_b - P_a)} e^{a+b} =: ? : ,$ and $\exp \left\{ -y [D(Q_b - Q_a) - B(P_b - P_a)]^2 \right\} =: ? : .$

In order to obtain the normally ordered expansion of $e^{D(Q_b - Q_a) - B(P_b - P_a)} e^{a+b}$, by noticing the eigenvalue equation of state $|\zeta\rangle$ (Eq.(28)) and the completeness relation of the state $|\zeta\rangle$ (Eq.(18)) as well as the expression of state $|\zeta\rangle$ (Eq.(19)), we have the operator identity

$$\begin{aligned} &e^{D(Q_b - Q_a) - B(P_b - P_a)} e^{a+b} \\ &= \int_{-\infty}^{\infty} dq \int \frac{d^2 z}{\pi} e^{\sqrt{2}q} e^{\sqrt{2}z} |z, q\rangle_{s, r, s, r} \langle z, q| \\ &=: \exp \left\{ [D(Q_b - Q_a) - B(P_b - P_a)] + \frac{D^2 + B^2}{2} + a + b \right\} :, \end{aligned} \quad (45)$$

where we have used the IWOP technique and the integration formulas

$$\int \frac{d^2 z}{\pi} e^{\xi |z|^2 + \eta z + \gamma z^*} = -\frac{1}{\xi} \exp \left[-\frac{\eta \gamma}{\xi} \right], \quad (46)$$

and

$$\int_{-\infty}^{\infty} \exp(-\alpha x^2 + \beta x) dx = \sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\beta^2}{4\alpha}\right). \quad (47)$$

Similarly, by using Eqs.(35,36), we also have

$$\begin{aligned} & e^{A(P_b - P_a) - C(Q_b - Q_a)} e^{a+b} \\ &= \int_{-\infty}^{\infty} dp \int \frac{d^2 z}{\pi} e^{A(P_b - P_a) - C(Q_b - Q_a)} e^{a+b} |\kappa_2\rangle \langle \kappa_2| \\ &= \int_{-\infty}^{\infty} dp \int \frac{d^2 z}{\pi} e^{\sqrt{2}p} e^{\sqrt{2}z} |\kappa_2\rangle \langle \kappa_2| \\ &=: \exp\left[a + b + \frac{(A^2 + C^2)}{2} + A(P_b - P_a) - C(Q_b - Q_a)\right], \end{aligned} \quad (48)$$

and

$$\begin{aligned} & \exp\left\{-y[D(Q_b - Q_a) - B(P_b - P_a)]^2\right\} \\ &= \int_{-\infty}^{\infty} dq \int \frac{d^2 z}{\pi} e^{-y(\sqrt{2}q)^2} |z, q\rangle_{s,r,s,r} \langle z, q| \\ &=: \sqrt{\frac{1}{1 + 2y(D^2 + B^2)}} \exp\left\{\frac{-y[D(Q_b - Q_a) - B(P_b - P_a)]^2}{1 + 2y(D^2 + B^2)}\right\}, \end{aligned} \quad (49)$$

as well as

$$\begin{aligned} & [D(Q_b - Q_a) - B(P_b - P_a)]^n \\ &= \int_{-\infty}^{\infty} dq \int \frac{d^2 z}{\pi} [D(Q_b - Q_a) - B(P_b - P_a)]^n |z, q\rangle_{s,r,s,r} \langle z, q| \\ &= \int_{-\infty}^{\infty} dq \int \frac{d^2 z}{\pi} [\sqrt{2}q]^n |z, q\rangle_{s,r,s,r} \langle z, q| \\ &=: \left(i\sqrt{\frac{D^2 + B^2}{2}}\right)^n H_n\left(\frac{D(Q_b - Q_a) - B(P_b - P_a)}{i\sqrt{2}\sqrt{D^2 + B^2}}\right), \end{aligned} \quad (50)$$

where we have used the generating function of Hermite polynomial

$$H_n(x) = \frac{\partial^n}{\partial t^n} \exp(2xt - t^2) \Big|_{t=0}. \quad (51)$$

Thus we can get some operator identities shown in Eqs.(45)-(51). These operator identities can be used in the quantum mechanics problems.

VI. CONCLUSION

In summary, we have proposed a new entangled state representation (the ICES representation $|z, q\rangle_{s,r}$) naturally from the completeness relations of the coherent state and the generalized coordinate state. We also have shown that mixing a symmetric beam splitter and the Fresnel transformation optical process can generate this new bipartite entangled state $|z, q\rangle_{s,r}$, which is a new project. We also discuss the new state's properties, i.e. the partly orthogonality and the eigenvalue equation. $|z, q\rangle_{s,r}$, also provides a way to predict a new squeezing operator and some new operator identities. These discussions not only exhibit the usefulness of the ICES representation, but also provides considerable insight into the character of them.

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